Classical and quantum 3 and 4-sieves to solve SVP with low memory

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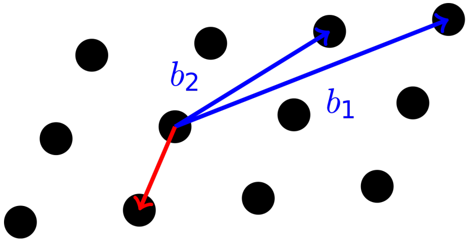
Lattice and SVP

Lattice

Given a basis $B = (\vec{b_1}, ..., \vec{b_d})$, the lattice $\mathcal{L}$ generated by B is the set of all integer linear combinations of its basis vectors: $\mathcal{L}(B) = \left\{ \sum_{i=1}^{d} z_i \vec{b_i},\ z_i \in \mathbb{Z} \right\}$.

Shortest Vector Problem (SVP)

Given a lattice $\mathcal{L}$, find the shortest non-zero vector $\vec{v} \in \mathcal{L}$.



Motivation to solve SVP

Cryptography

- NP-hard problem, hard in average, believed to be quantum-resistant.
- Problems derived from SVP: LWE, SIS, NTRU...
- Cryptosystems based on them: Kyber, Dilithium, Falcon (NIST standardization), FHE

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Cryptanalysis

- Broken if we can find a reduced basis of the lattice.
- BKZ algorithm returns a reduced basis using an SVP-solver.

$\Rightarrow$ The security of these cryptosystems directly relies on the complexity of solving SVP.

Overview

1. Lattice sieving
   Configuration problem
2. Filtering
   New Random Product Code for filtering
3. Framework to solve SVP
4. Trade-offs for classic and quantum k-sieves

Sieving

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Heuristic: Lattice vectors act as random vectors.

- Implies that vectors of norm at most R are w.h.p. of norm very close to R.
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Input: list L of N lattice vectors of norm at most R ; $\gamma < 1$.
Output: list L_{out} of N lattice vectors of norm at most $\gamma R < R$.

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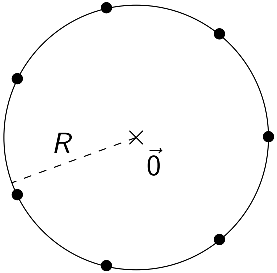
Initialization:
Generate N lattice vectors
of norm $\leq R$
(Klein's algorithm)



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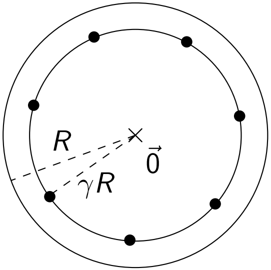
After 1 iteration:
vectors of norm at most γR



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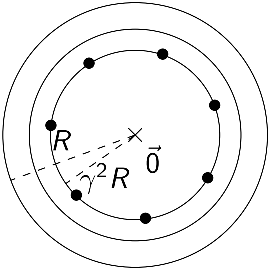
After 2 iterations:
vectors of norm at most $\gamma^2 R$



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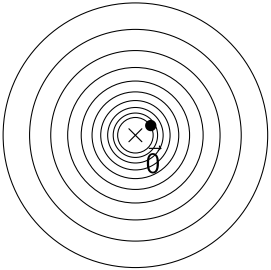
After poly(d) iterations:
norm at most $\gamma^{\text{poly}(d)} R$.

Short vector found!

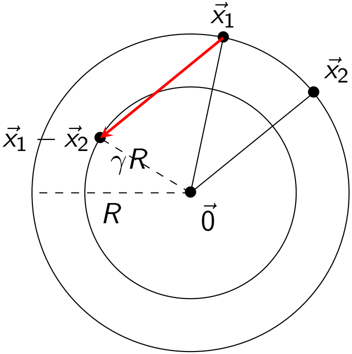


Sieving step

Nguyen-Vidick sieve [NV08] (2-sieve)

for $(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \in L \times L$:
  if $\|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2\| \leqslant \gamma R$:
    add $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ to L_{out}

Sphere of dimension d and radius R:



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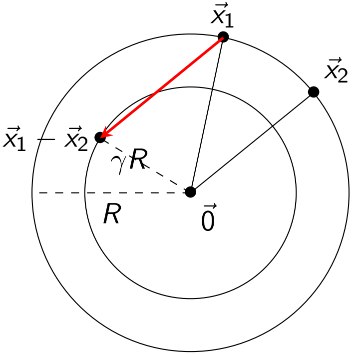

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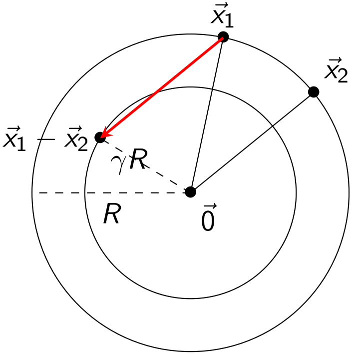

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Condition of reduction:
For $\gamma = 1$, $\|\vec{\mathbf{x}}_1\| = \|\vec{\mathbf{x}}_2\| = R$,

$$\|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2\| \leqslant \gamma R$$
$$\Leftrightarrow \text{Angle}(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \leqslant \frac{\pi}{3}$$

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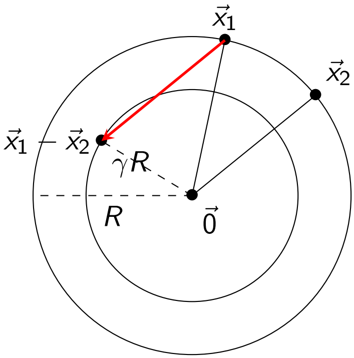

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$$\|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2\| \leqslant \gamma R$$
$$\Leftrightarrow \text{Angle}(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \leqslant \frac{\pi}{3}$$
$$\Leftrightarrow \frac{1}{R^2}\langle \vec{\mathbf{x}}_1 | \vec{\mathbf{x}}_2 \rangle \geq \frac{1}{2}.$$

Sieving step

3-sieve

for $(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3) \in L^3$:
  if $\|\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 + \vec{\mathbf{x}}_3\| \leqslant \gamma R$:
    add $\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 + \vec{\mathbf{x}}_3$ to L_{out}

4-sieve

for $(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \vec{\mathbf{x}}_4) \in L^4$
  if $\|\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 + \vec{\mathbf{x}}_3 + \vec{\mathbf{x}}_4\| \leqslant \gamma R$:
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k-sieve

for $(\vec{\mathbf{x}}_1, ..., \vec{\mathbf{x}}_k) \in L^k$
  if $\|\vec{\mathbf{x}}_1 + ... + \vec{\mathbf{x}}_k\| \leqslant \gamma R$:
    add $\vec{\mathbf{x}}_1 + ... + \vec{\mathbf{x}}_k$ to L_{out}

Minimal size of the list L

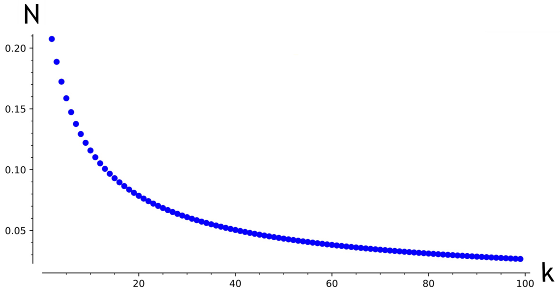
Sieving step

Input: List of N lattice vectors
Output: List of N reduced lattice vectors

$\Rightarrow$ We need that there exists N reduced vectors calculable from the N input vectors.

Notation: $2^{xd+o(d)}$

k	Memory N	Time (naive) N^k
2	0.208	0.415
3	0.189	0.566
4	0.173	0.690
5	0.159	0.794
6	0.147	0.884

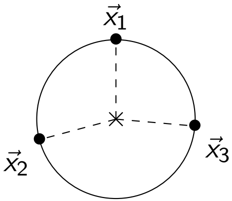


Reduction to the configuration problem

Configurations

Configuration

k-tuple $(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k)$ satisfies configuration $C = (C_{ij})_{i,j} \in \mathbb{R}^{k \times k}$ iff. $\langle \vec{\mathbf{x}}_i | \vec{\mathbf{x}}_j \rangle \leq C_{ij}$ (with $C_{ij} \leq 0$).



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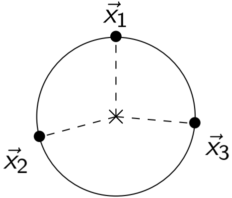

Valid configuration C: $(\vec{\mathbf{x}}_1, ..., \vec{\mathbf{x}}_k)$ satisfies $C \Rightarrow \|\vec{\mathbf{x}}_1 + ... + \vec{\mathbf{x}}_k\| \leq \gamma R$

Configurations

Configuration problem

Input: List L, a valid configuration C
Output: Tuples $(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k)$ for $\vec{\mathbf{x}}_i \in L$ satisfying configuration C

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***k*-sieve problem**

Input: List L
Output: Vectors $\sum_{i=1}^{k} \vec{\mathbf{x}}_i$ for $\vec{\mathbf{x}}_i \in L$ of norm $\leq \gamma R$.

Configurations

Configuration problem

Input: Lists $L_1, \ldots, L_k$, a valid configuration C
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$\Rightarrow$

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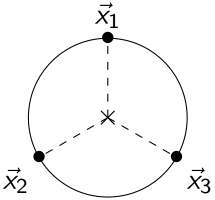
Configurations

Balanced configuration

- Fix $C_{ij} = -1/k$ for $i \neq j$
- The most common configuration for reducing k-tuples
  $\Rightarrow$ Minimizes the memory $|L|$.



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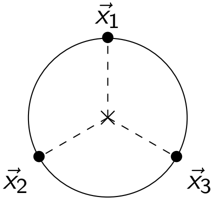
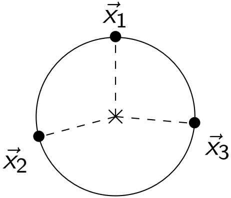


Any configuration

- Only constraint: $\|\vec{\mathbf{x}}_1 + ... + \vec{\mathbf{x}}_k\| \leq \gamma R$
- Rarer configurations $\Rightarrow$ Require longer list, but the tuples can be easier to find.



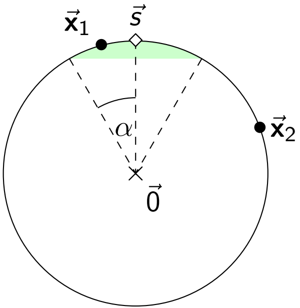
Locality Sensitive Filtering (LSF)

Filtering

Locality Sentitive Filter

A **filter** $f_{\vec{s},\alpha}$ of center $\vec{s} \in \mathbb{R}^d$ and angle $\alpha \in [0, \pi/2]$ maps a vector $\vec{x}$ to a boolean value:

- 1 if Angle$(\vec{x}, \vec{s}) \leqslant \alpha$,
- 0 else.



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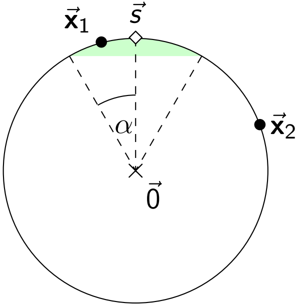


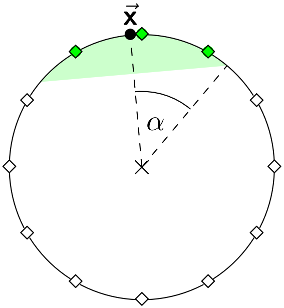
Each filter is associated with a set that we can fill with vectors.

Filtering - Random Product Code

Random Product Code (RPC) of parameters $[d, m, B]$

$$\mathfrak{C} = Q \cdot (\mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m) \subset \mathbb{R}^d$$

- $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$: sets of B vectors in $\mathbb{R}^{d/m}$ sampled unif. & indep. random of norm $\sqrt{1/m}$
- Q uniformly random rotation over $\mathbb{R}^d$



Codewords $\diamond$

- Uniformly distributed over the sphere
- Each codeword = center of one filter
- Decode $\vec{\mathbf{x}}$ in efficient time (subexp. or poly)

Filtering - Random Product Code

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List Decoding Algorithm for RPC [BDGL16]

Input: Random Product Code $\mathfrak{C}$, vector $\vec{\mathbf{x}}$, angle α

Output: List of all the filters $\mathbf{F} \in \mathfrak{C}$ of angle at most α with $\vec{\mathbf{x}}$.

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4. Assemble the obtained codewords of $\mathfrak{C}_1, ..., \mathfrak{C}_m$
5. Apply rotation Q to recover $\mathfrak{C}$'s codewords = nearest filters of $\vec{\mathbf{x}}$

Filtering - Solving SVP

2-sieve

For each vector: search a reducing vector within the whole list L.

2-sieve with filtering

1. Generate the filters $\triangleright$ Sample a RPC
2. Add each vector to its filters of angle at most α. $\triangleright$ List decoding algorithm
3. For each vector : search a reducing vector within its filters.

Filtering - Solving SVP

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1. Generate the filters ▷ Sample a RPC
2. Add each vector to its filters of angle at most α. ▷ List decoding algorithm
3. For each vector : search a reducing vector within its filters.
   - Classically or by Grover's search

Time complexity (for minimal memory $N = 2^{0.208d+o(d)}$):

Classical 2-sieve: $2^{0.415d+o(d)}$

With filtering: $2^{0.292d+o(d)}$

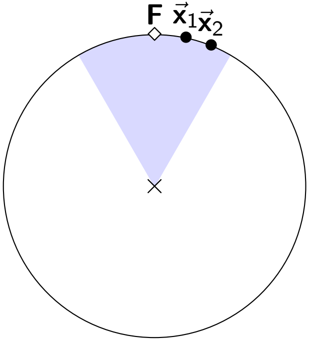
Quantum 2-sieve: $2^{0.312d+o(d)}$

With filtering: $2^{0.265d+o(d)}$

New code for filtering

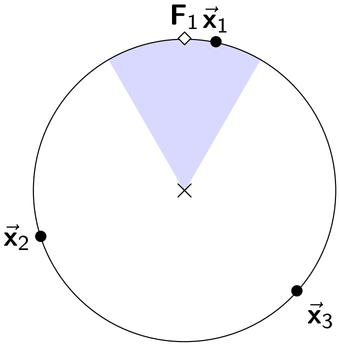
Filtering strategy for the 2-sieve

Constraint: $\langle \vec{\mathbf{x}}_1 | \vec{\mathbf{x}}_2 \rangle \geq \frac{1}{2}$



Filtering strategy for the k-sieve

Constraints: $\langle \vec{\mathbf{x}}_i | \vec{\mathbf{x}}_j \rangle \leq C_{ij}$



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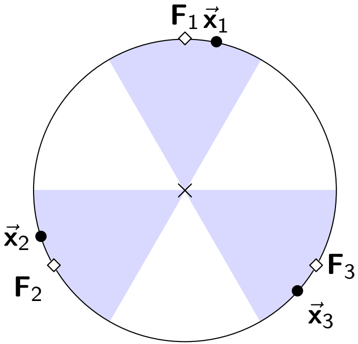

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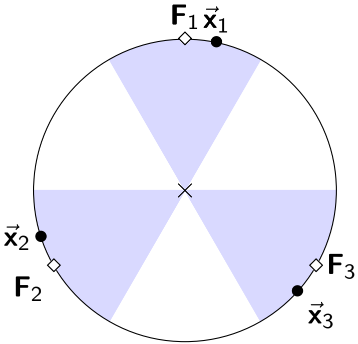


k-Random Product Code

A k-RPC $\mathfrak{C}$ is a code such that

$$\forall \mathbf{F}_1 \in \mathfrak{C}, \exists \mathbf{F}_2, \ldots, \mathbf{F}_k \in \mathfrak{C} \text{ st. } \sum_{i=1}^{k} \mathbf{F}_i = \vec{0}.$$

Framework for the k-sieve

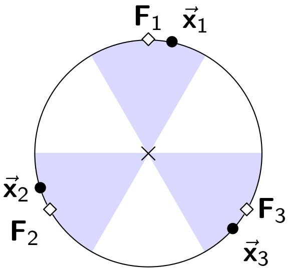
Framework

k-sieve framework to solve SVP

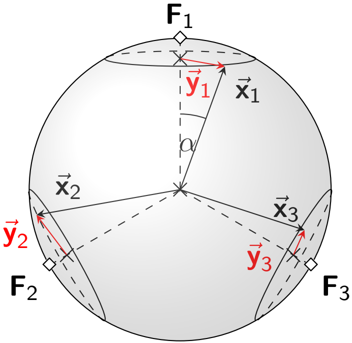
Input: list L of N lattice vectors, parameters k, angle α, configuration C

Output: list L_{out} of N reduced lattice vectors

1. Generate the tuple-filters. **Prefilter** L: for each $\vec{\mathbf{x}} \in L$, add $\vec{\mathbf{x}}$ to its nearest (unique) filter.
2. For each tuple-filter: **Find all solutions** satisfying C within the tuple-filter.
3. Repeat 1. and 2. until $|L_{out}| = N$.



Residual vectors



Search for a tuple $(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k)$ satisfying configuration C	$\Leftrightarrow$	Search for their residual vectors $(\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_k)$ satisfying configuration $C'_{C,\alpha}$

Framework

k-sieve framework to solve SVP

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3. Repeat 1. and 2. until $|L_{out}| = N$.

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k-sieve framework to solve SVP

Input: list L of N lattice vectors, parameters k, angle α, configuration C
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Subroutine **Find All Solutions** within a tuple-filter

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$$T(k\text{-sieve}) := \left(|L|_C + NbFilters_\alpha \cdot T(\mathbf{FAS}_{C'_{C,\alpha}})\right) \cdot NbRep_{C,\alpha}$$

Subroutine "Find All Solutions"

For $k = 2$:

- 2-sieve via quantum random walks [CL21, BCSS23]

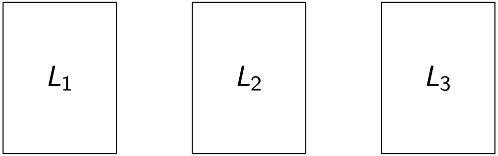
$k = 3$:

- Classic 3-sieve
- Quantum 3-sieve

$k = 4$:

- Classic 4-sieve
- Quantum 4-sieve

Classic 3-sieve — Subroutine



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$(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3)$ satisfies C'

$$\Leftrightarrow \begin{cases} \langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_2 \rangle & \leq C'_{12} \\ \langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_3 \rangle & \leq C'_{13} \\ \langle \vec{\mathbf{y}}_2 | \vec{\mathbf{y}}_3 \rangle & \leq C'_{23} \end{cases}$$

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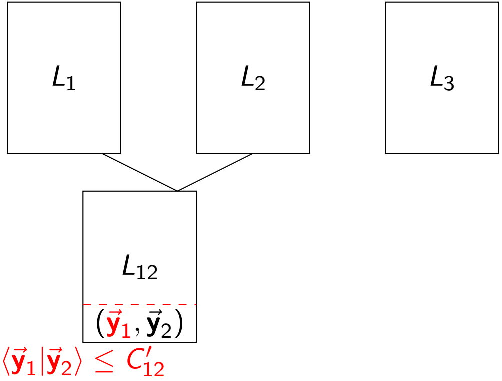

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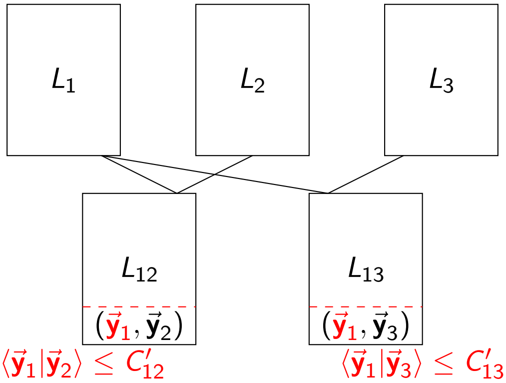

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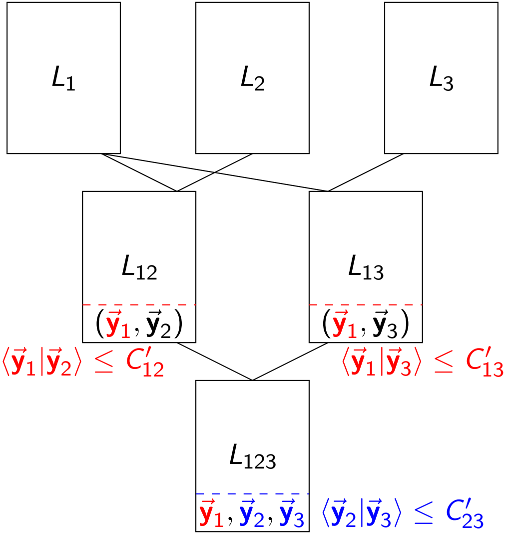
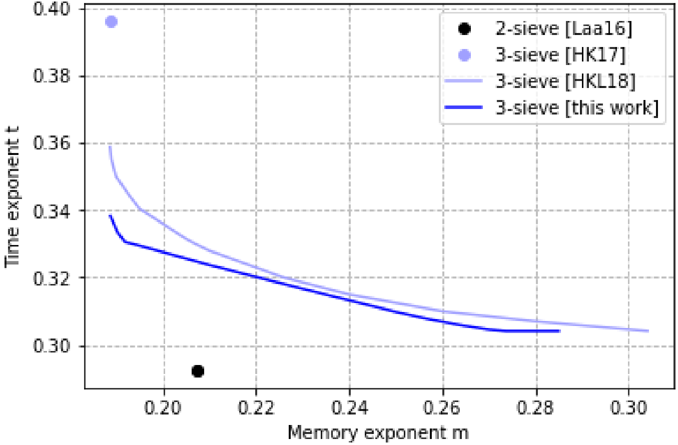


Classic 3-sieve

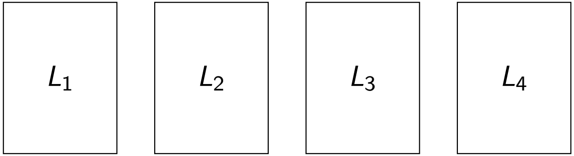


Classic 4-sieve — Subroutine

Configuration problem

Input: Lists L_1, L_2, L_3, L_4, configuration C'

Output: All the tuples $(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \vec{\mathbf{y}}_4) \in L_1 \times L_2 \times L_3 \times L_4$ satisfying configuration C'



$(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \vec{\mathbf{y}}_4)$ satisfies C'

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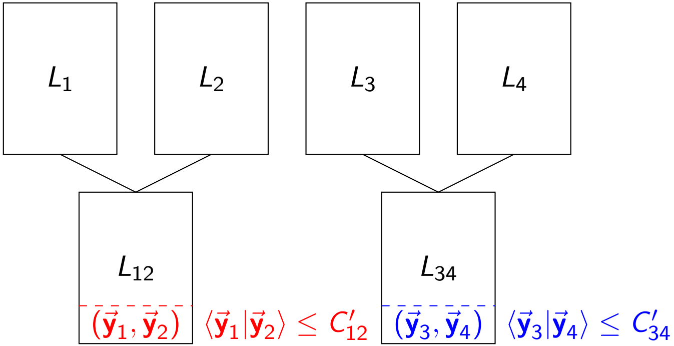

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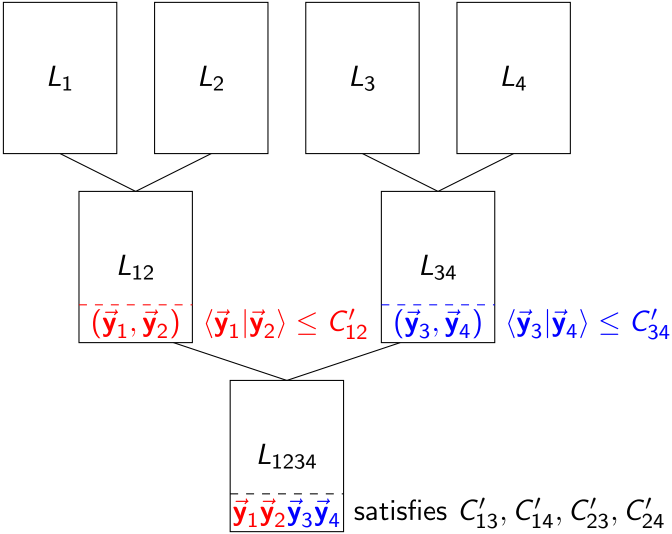
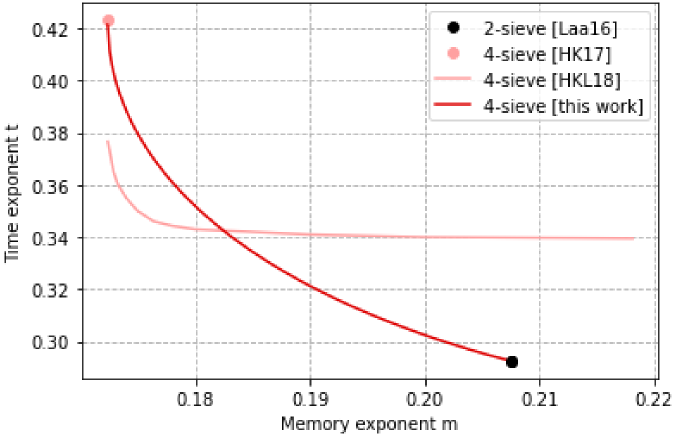
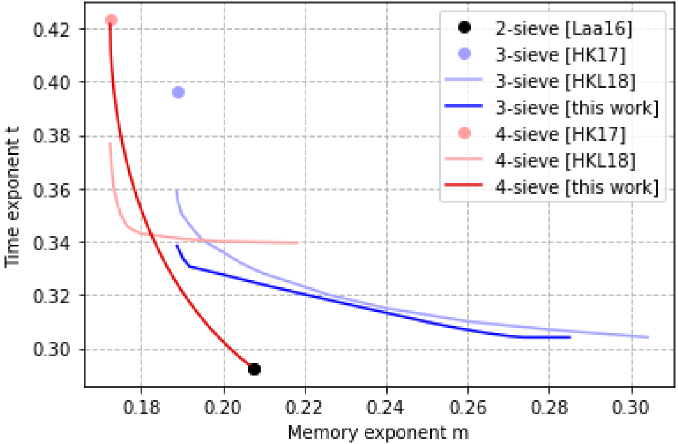


Classic 4-sieve



Classic k-sieves



Quantum 3-sieve — Subroutine

$$|\psi_{L_1}\rangle \qquad |\psi_{L_2}\rangle \qquad |\psi_{L_3}\rangle$$

Quantum 3-sieve — Subroutine

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$$\|$$

$$\frac{1}{\sqrt{|L_1|}} \sum_{\vec{\mathbf{y}}_1 \in L_1} |\mathrm{i}_{\vec{\mathbf{y}}_1}\rangle |\vec{\mathbf{y}}_1\rangle$$

Quantum 3-sieve — Subroutine

$$|\psi_{L_1}\rangle \qquad |\psi_{L_2}\rangle \qquad |\psi_{L_3}\rangle$$

$$\| \qquad \Big\downarrow \text{Grover}$$

$$\frac{1}{\sqrt{|L_1|}} \sum_{\vec{\mathbf{y}}_1 \in L_1} |\mathsf{i}_{\vec{\mathbf{y}}_1}\rangle |\vec{\mathbf{y}}_1\rangle \qquad |\psi_{L_2(\vec{\mathbf{y}}_1)}\rangle$$

$$\langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_2 \rangle \leq C'_{12}$$

Quantum 3-sieve — Subroutine

$$
\begin{array}{ccc}
|\psi_{L_1}\rangle & |\psi_{L_2}\rangle & |\psi_{L_3}\rangle \\
\| & \big\downarrow \text{Grover} & \big\downarrow \text{Grover} \\
\frac{1}{\sqrt{|L_1|}} \sum_{\vec{\mathbf{y}}_1 \in L_1} |\mathrm{i}_{\vec{\mathbf{y}}_1}\rangle |\vec{\mathbf{y}}_1\rangle & |\psi_{L_2(\vec{\mathbf{y}}_1)}\rangle & |\psi_{L_3(\vec{\mathbf{y}}_1)}\rangle
\end{array}
$$

$$\langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_2 \rangle \leq C'_{12}$$
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$$\langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_2 \rangle \le C'_{12}$$
$$\langle \vec{\mathbf{y}}_1 | \vec{\mathbf{y}}_3 \rangle \le C'_{13}$$

$$\|$$

$$\frac{1}{\sqrt{|L_2(\vec{\mathbf{y}}_1)|}} \sum_{\vec{\mathbf{y}}_2 \in L_2(\vec{\mathbf{y}}_1)} |\mathrm{i}_{\vec{\mathbf{y}}_2}\rangle |\vec{\mathbf{y}}_2\rangle$$

Quantum 3-sieve — Subroutine

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$$\| \qquad \downarrow \text{Grover} \qquad \downarrow \text{Grover}$$

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$$\| \qquad \downarrow \text{Grover}$$

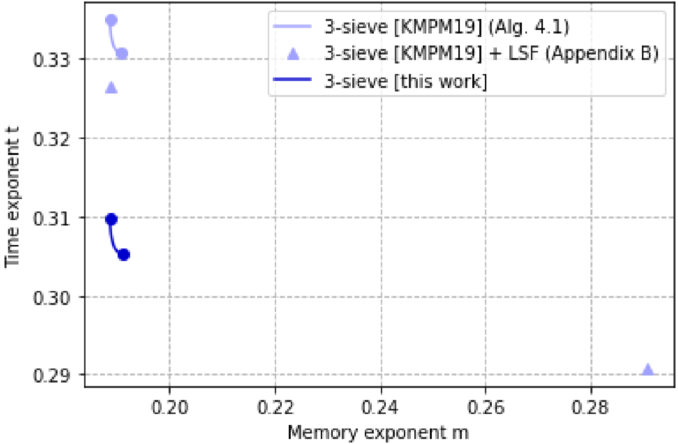
$$\frac{1}{\sqrt{|L_2(\vec{\mathbf{y}}_1)|}} \sum_{\vec{\mathbf{y}}_2 \in L_2(\vec{\mathbf{y}}_1)} |i_{\vec{\mathbf{y}}_2}\rangle |\vec{\mathbf{y}}_2\rangle \qquad |\psi_{L_3(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2)}\rangle \qquad \langle \vec{\mathbf{y}}_2 | \vec{\mathbf{y}}_3 \rangle \leq C'_{23}$$

Quantum 3-sieve — Subroutine

$$|\psi_{L_1}\rangle|\psi_{L_2(\vec{\mathbf{y}}_1)}\rangle|\psi_{L_3(\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2)}\rangle$$

- Apply amplitude amplification
- Measure and get a reducing $(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3)$
- Repeat to find all the solutions in $L_1 \times L_2 \times L_3$

Quantum 3-sieve



Quantum 4-sieve — Subroutine

$$|\psi_{L_1}\rangle|\psi_{L_2(\vec{\mathbf{y}}_1)}\rangle|\psi_{L_3(\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2)}\rangle|\psi_{L_4(\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2)}\rangle$$

- Apply amplitude amplification
- Measure and get a reducing $(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \vec{\mathbf{y}}_4)$
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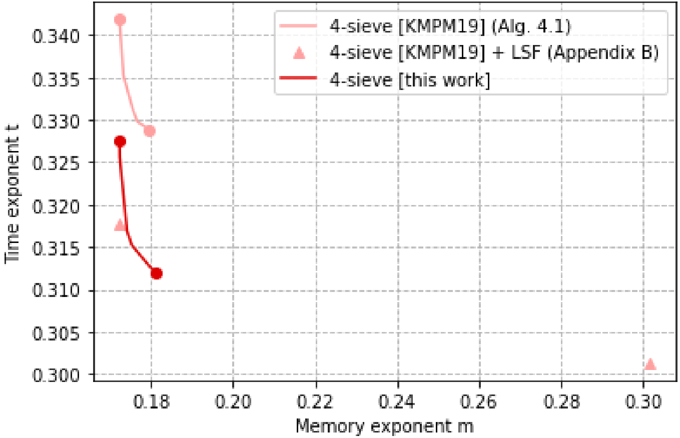
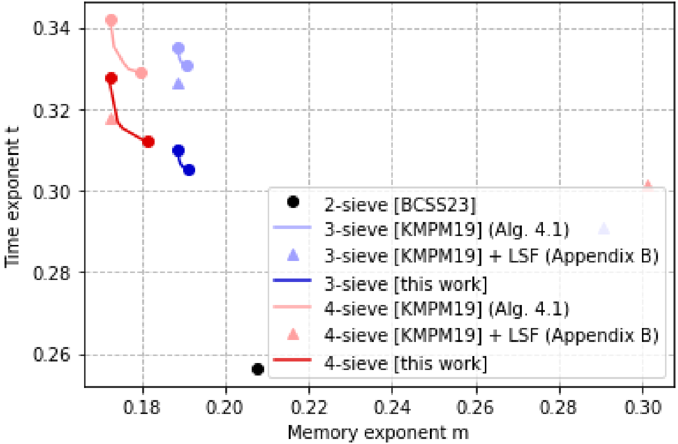
Quantum 4-sieve



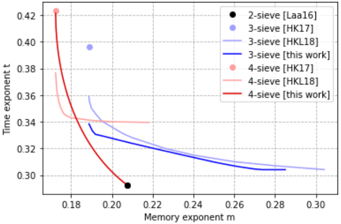
Quantum k-sieves

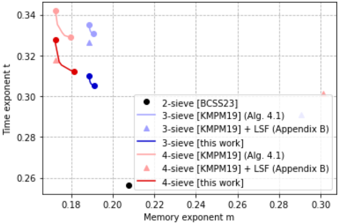


Conclusion

This work:

- Improves the 3-sieves trade-off
- New trade-offs for the 4-sieves





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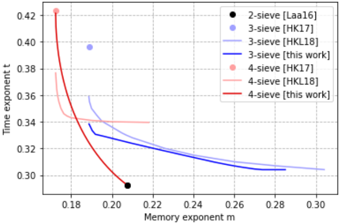

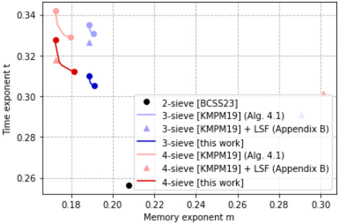

Further research:

- k-sieve for $k > 4$
- Mix our prefiltering with inner filtering as in [HKL18, KMPM19]
- **Classical**: Optimal merging trees
- **Quantum**: k-sieve via quantum random walks

Thank you for listening!
Any questions?

References

[NV08] P.Q. Nguyen and T. Vidick
Sieve algorithms for the shortest vector problem are practical
doi.org/10.1515/JMC.2008.009

[BDGL16] A. Becker, L. Ducas, N. Gama and T. Laarhoven
New directions in nearest neighbor searching with applications to lattice sieving
ePrint 2015/1128

[HKL18] G. Herold, E. Kirshanova and T. Laarhoven (2018)
Speed-ups and time–memory trade-offs for tuple lattice sieving
ePrint 2017/1228

[KMPM19] E. Kirshanova, E. Mårtensson, E.W. Postlethwaite and S.R. Moulik
Quantum algorithms for the approximate k-list problem and their application to lattice sieving
ePrint 2019/1016

[this work] A. Chailloux and J. Loyer
Classical and quantum 3 and 4-sieves to solve SVP with low memory
ePrint 2023/200

k-Random Product Code (Technical details)

Sample a RPC $\mathfrak{C} = Q \cdot (\mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m)$

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For each $F_1 \in \mathfrak{C}$:

$k = 3$

$\mathbf{F}_1$	$\in \mathfrak{C}$

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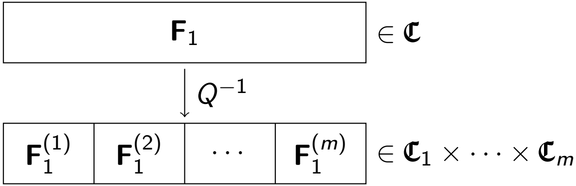

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$\mathbf{F}_1$

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$\downarrow Q^{-1}$

$\mathbf{F}_1^{(1)}$	$\mathbf{F}_1^{(2)}$	$\cdots$	$\mathbf{F}_1^{(m)}$

$\in \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m$

Choose a random

$\mathbf{F}_2^{(1)}$	$\mathbf{F}_2^{(2)}$	$\cdots$	$\mathbf{F}_2^{(m)}$

such that for $i \in [m], \langle \mathbf{F}_1^{(i)} | \mathbf{F}_2^{(i)} \rangle = -\frac{1}{2m}$.

k-Random Product Code (Technical details)

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For each $F_1 \in \mathfrak{C}$:

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$\mathbf{F}_1$	$\in \mathfrak{C}$

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$\mathbf{F}_1^{(1)}$	$\mathbf{F}_1^{(2)}$	$\cdots$	$\mathbf{F}_1^{(m)}$	$\in \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m$

Choose a random	$\mathbf{F}_2^{(1)}$	$\mathbf{F}_2^{(2)}$	$\cdots$	$\mathbf{F}_2^{(m)}$	such that for $i \in [m]$, $\langle \mathbf{F}_1^{(i)} \vert \mathbf{F}_2^{(i)} \rangle = -\frac{1}{2m}$.

$\downarrow Q$

$\mathbf{F}_2$	with $\langle \mathbf{F}_1 \vert \mathbf{F}_2 \rangle = -\frac{1}{2}$.

Compute $\mathbf{F}_3 = -\mathbf{F}_1 - \mathbf{F}_2$.

k-Random Product Code (Technical details)

Sample a RPC $\mathfrak{C} = Q \cdot (\mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m)$

For each $F_1 \in \mathfrak{C}$:

$\forall k$

$$\boxed{\mathbf{F}_1} \in \mathfrak{C}$$

$$\downarrow Q^{-1}$$

$$\boxed{\mathbf{F}_1^{(1)}}\boxed{\mathbf{F}_1^{(2)}}\boxed{\cdots}\boxed{\mathbf{F}_1^{(m)}} \in \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_m$$

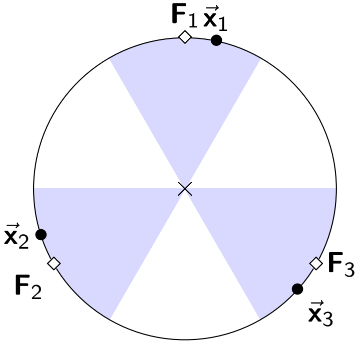
For $j = 2 \ldots k-1$, choose random $\boxed{\mathbf{F}_j^{(1)}}\boxed{\mathbf{F}_j^{(2)}}\boxed{\cdots}\boxed{\mathbf{F}_j^{(m)}}$ st. for $i \in [m], j' < j, \langle \mathbf{F}_{j'}^{(i)} | \mathbf{F}_j^{(i)} \rangle = -\frac{1}{(k-1)m}$.

$$\downarrow Q$$

$$\boxed{\mathbf{F}_j}$$ with $\langle \mathbf{F}_1 | \mathbf{F}_2 \rangle = -\frac{1}{k-1}$.

Compute $\mathbf{A}_k = -\sum_{j=1}^{k-1} \mathbf{F}_j$.

Filtering strategy for the k-sieve



Tuple-filter
$(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$

k-Random Product Code

A k-RPC $\mathfrak{C}$ is a code such that

$$\forall \mathbf{F}_1 \in \mathfrak{C}, \exists \mathbf{F}_2, ..., \mathbf{F}_k \in \mathfrak{C} \text{ st. } \sum_{i=1}^{k} \mathbf{F}_i = \vec{0}.$$

With an **efficient** decoding algorithm