Lattice sieving via quantum random walks

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Joint work with André Chailloux

Overview

1. Preliminaries
   - Lattices
   - Locality sensitive filtering (LSF)
   - Quantum Computing
2. Our algorithm
3. Complexity and space/time trade-offs

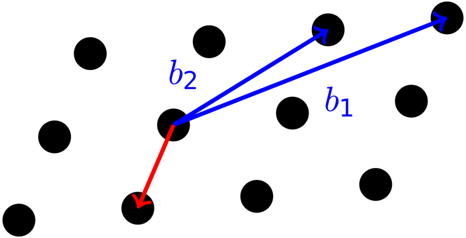
Lattice and SVP

Lattice

The d-dimensional lattice $\mathcal{L} \subset \mathbb{R}^m$ generated by the basis $B = (\vec{b_1}, \ldots, \vec{b_d})$ with $\forall i, \vec{b_i} \in \mathbb{R}^m$ is the set of all integer linear combinations of its basis vectors: $\mathcal{L}(B) = \left\{ \sum_{i=1}^{d} \lambda_i \vec{b_i},\ \lambda_i \in \mathbb{Z} \right\}$.

Shortest Vector Problem (SVP)

Given a lattice $\mathcal{L}$, find the shortest non-zero vector $\vec{v} \in \mathcal{L}$, ie. st. $||\vec{v}|| = \inf\left\{ ||\vec{u}|| \neq 0,\ \vec{u} \in \mathcal{L} \right\}$.



Why do we want to solve SVP?

Cryptography

- NP-hard problem, hard in average.
- Problems derived from SVP: SIS, LWE, NTRU...
- Quantum-resistant cryptosystems based on them: Dilithium, FALCON, NTRU, Kyber, SABER.

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Cryptanalysis

- Broken if a reduced basis of the lattice can be found.
- BKZ algorithm finds a reduced basis.
- Solving SVP = subroutine of BKZ

$\Rightarrow$ The security of these cryptosystems directly relies on the complexity of solving SVP.

Sieving

SVP-solving methods

- Main practical methods: enumeration and sieving.
- Run in exponential time.

Main heuristic: Lattice vectors acts as random vectors.

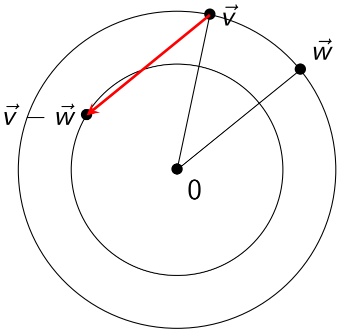
- Implies that vectors of norm at most R are lying on the border of $R \cdot \mathcal{S}^d$, with $R \cdot \mathcal{S}^d := \{\vec{x} \in \mathbb{R}^d : \|\vec{x}\| \leqslant R\}$.
- Validated by experiments.

Sieving

Nguyen-Vidick Sieve (NV-sieve) [NV08]

Input: list L of N lattice vectors of norm at most R ; $\gamma < 1$.
Output: list L' of N lattice vectors of norm at most $\gamma R < R$.
for $(\vec{v}, \vec{w}) \in L$:
    if $\|\vec{v} - \vec{w}\| \leqslant \gamma R$: add $\vec{v} - \vec{w}$ to L'

Sphere of dimension d and radius R.



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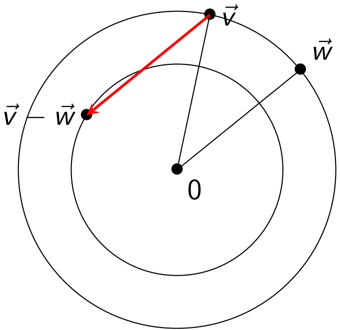


If $\vec{v}, \vec{w} \in \mathcal{L}$ then $\vec{v} - \vec{w} \in \mathcal{L}$.

For $\gamma \to 1$ and $\vec{v}, \vec{w} \in R \cdot \mathcal{S}^d$, $\|\vec{v} - \vec{w}\| \leqslant \gamma R \Leftrightarrow \theta(\vec{v}, \vec{w}) \leqslant \frac{\pi}{3}$.

Sieving - Solving SVP

Solve SVP by sieving

Input: a lattice $\mathcal{L}$ of basis $(\vec{b}_1, ..., \vec{b}_d)$
Output: a shortest vector of $\mathcal{L}$ (probably)
$L \leftarrow$ generate $N = (4/3)^{d/2+o(d)}$ lattice vectors ▷ by Klein's algorithm
while L does not contain a short vector :
  $L \leftarrow$ **NV-sieve step**$(L, \gamma \to 1)$
return min(L)

1st iteration: norm γR
2nd iteration: $\gamma^2 R$
⋮
poly(d)-th iteration: $\gamma^{poly(d)} R$

Complexity: $N^2 = 2^{0.415d+o(d)}$ time and $N = 2^{0.208d+o(d)}$ space.

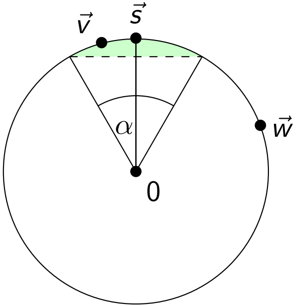
LSF (Locality Sensitive Filtering)

Improvement of the NV-sieve: only check pairs of close vectors.

Filter

A **filter** $f_{\vec{s},\alpha}$ of center $\vec{s} \in \mathbb{R}^d$ and angle $\alpha \in [0, \pi/2]$ maps a vector $\vec{v}$ to a boolean value:

- 1 if $\theta(\vec{v}, \vec{s}) \leqslant \alpha$,
- 0 else.



LSF (Locality Sensitive Filtering)

NV-sieve with LSF

1. Generate filters all over the sphere. $\triangleright$ centers $=$ words from a code
2. Add each vector to its nearest filters of angle at most α. $\triangleright$ list decoding algorithm
3. For each vector : search a reducing one within its filters (instead of in the whole list).
   - Classically or by Grover's search

Complexity ($2^{0.208d+o(d)}$ space):
Original NV-sieve [NV08]: $2^{0.415d+o(d)}$ time.
Classic with LSF [BDGL16]: $2^{0.292d+o(d)}$ time.
Quantum with LSF [Laa16]: $2^{0.265d+o(d)}$ time.

Grover's algorithm

Input: $x_1, ..., x_n \in E^d$ and a function $f : E^d \to \{0, 1\}$.
Output: $i \in [|1, n|]$ such that $f(x_i) = 1$.

Time complexity: $O(\sqrt{n})$.

Quantum Random Walk

Input: a graph $G = (V, E)$,
a function $f : V \to \{0, 1\}$ with $f(v) = 1 \Leftrightarrow v$ is a "**marked**" vertex.
Output: a marked vertex $v \in V$.

(Will be illustrated further with an example.)

Our algorithm

NV-sieve using quantum random walks

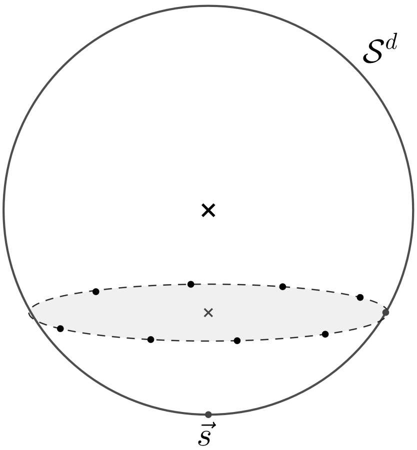
Input: list L of N lattice vectors of norm at most R ; $\gamma < 1$.
Output: list L' of N lattice vectors of norm at most $\gamma R < R$.

Main idea: Replace Grover's search by a quantum random walk.

Step 1

Sample a code C and generate the α-filters.

Insert each list vector in its (unique) nearest α-filter. $\triangleright$ N^{c_α} vectors per α-filter. $c_\alpha \in [0, 1]$



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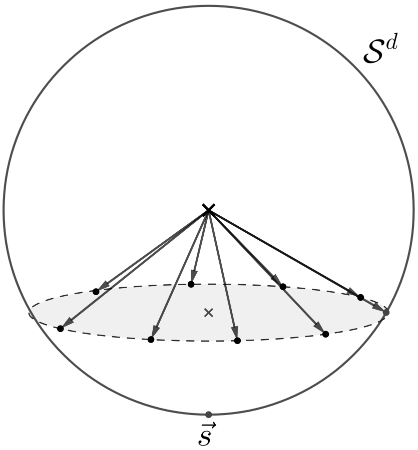

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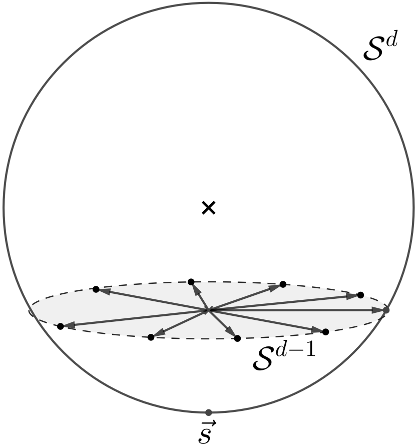

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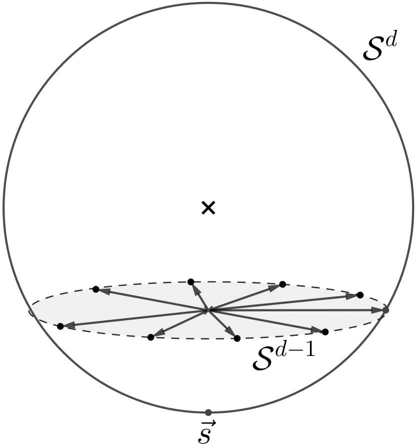


For $\vec{v}, \vec{w} \in \mathcal{S}^d$ and their residual vectors $\vec{v}_R, \vec{w}_R \in \mathcal{S}^{d-1}$,

$$\theta(\vec{v}, \vec{w}) \leqslant \frac{\pi}{3} \Leftrightarrow \theta(\vec{v}_R, \vec{w}_R) \leqslant \theta^*_\alpha.$$

Step 2

For each α-filter :

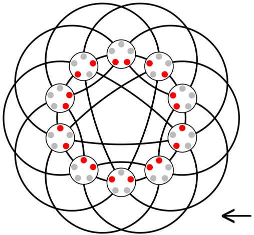
1. Vertex : Choose randomly N^{c_V} vectors from the α-filter.
2. Sample a code C' and generate the β-filters.
   Insert each Vertex's vector in its nearest β-filter.
3. Perform Quantum Random Walks to find all the reducing pairs in the α-filter.

Quantum Random Walk

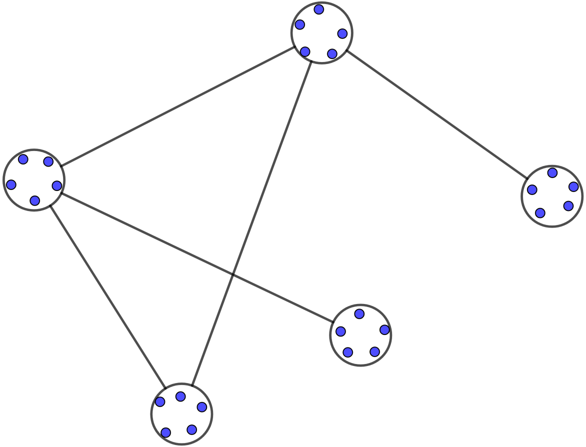
Input: a graph $G = J(N^{c_\alpha}, N^{c_V})$,
a vertex is marked iff. contains a pair of angle at most θ^*_α.
Output: a marked vertex.

Johnson's graph $J(N^{c_\alpha}, N^{c_V})$:

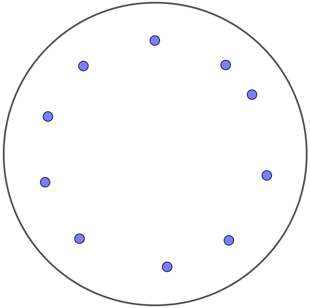
- Vertexes V: set of N^{c_V} from the N^{c_α} of the current α-filter.
- Edges E: 2 vertexes are neighbors iff. they differ by exactly 1 vector.

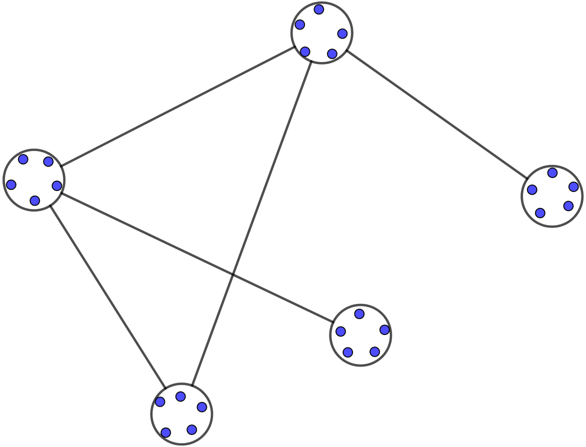


← Johnson's graph $J(5,2)$



Zoom on the current vertex





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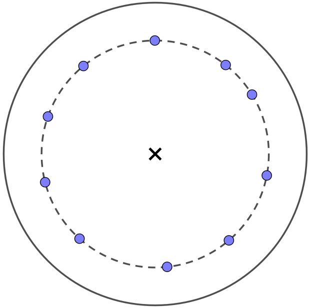

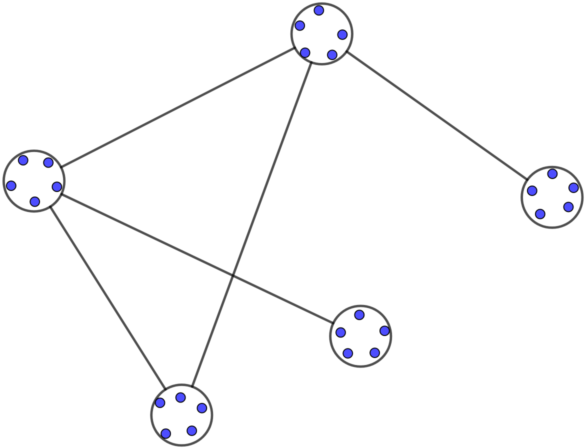

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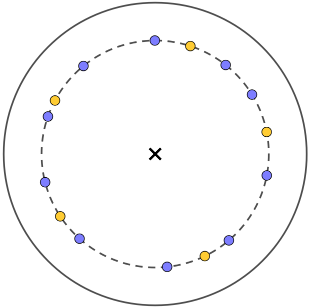

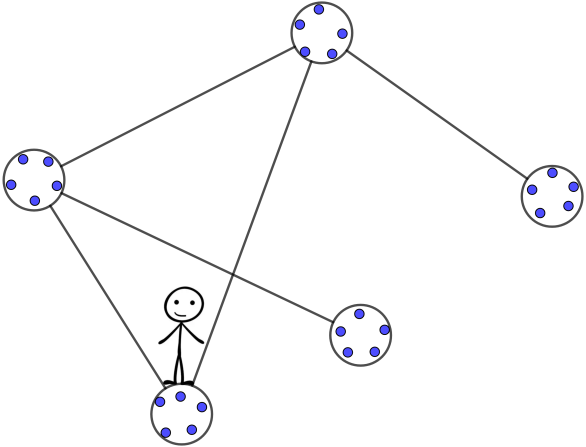

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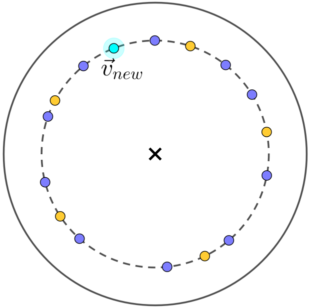

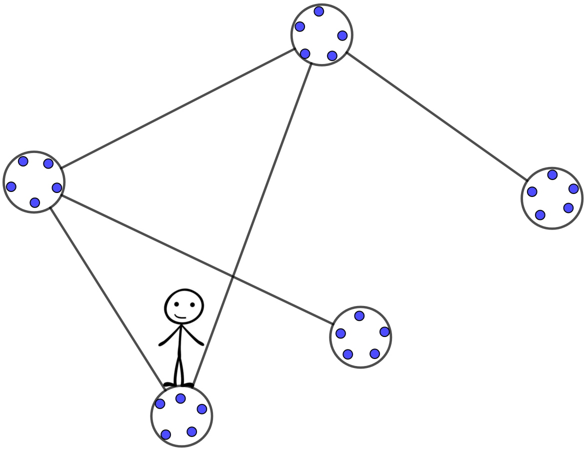

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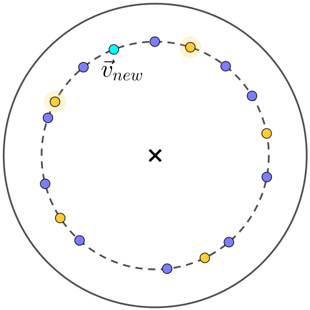

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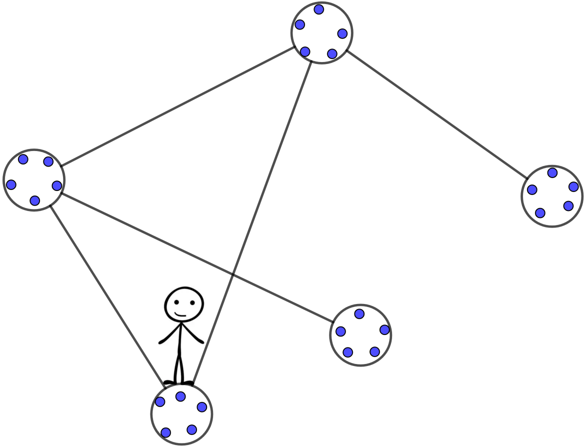

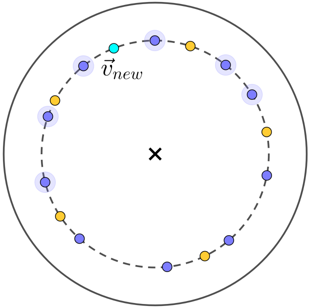

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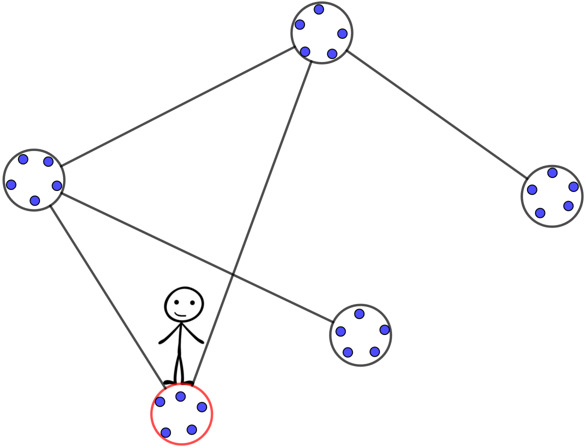

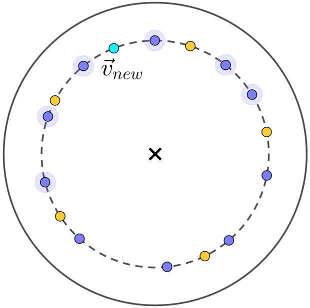

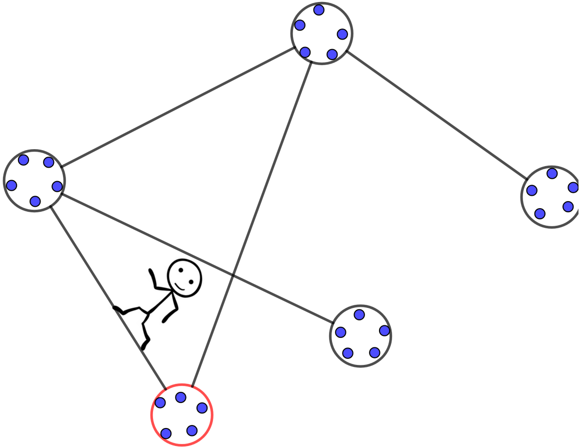

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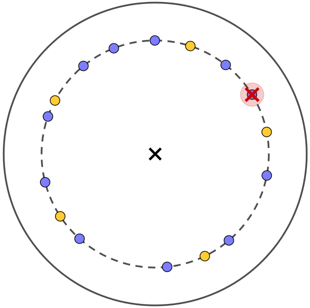

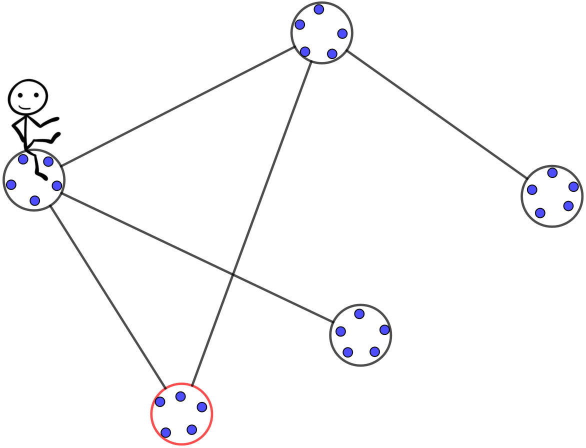

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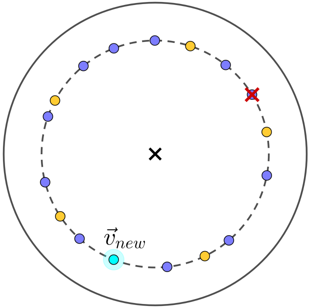

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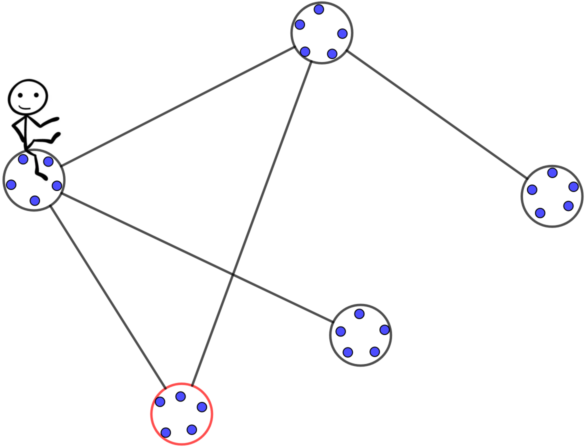

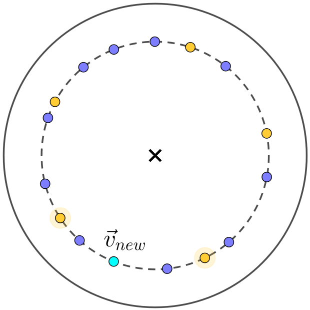

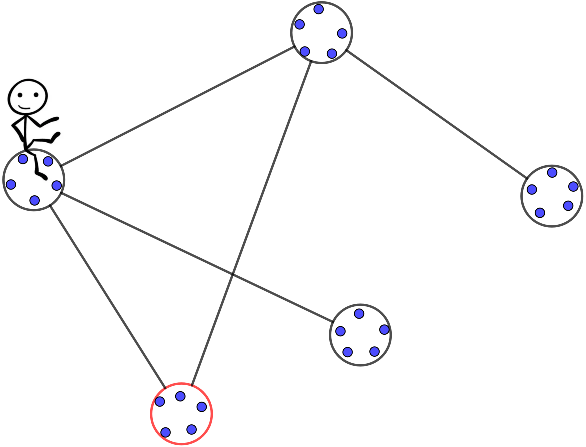

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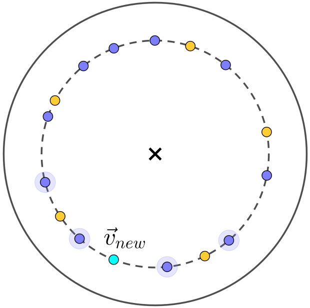

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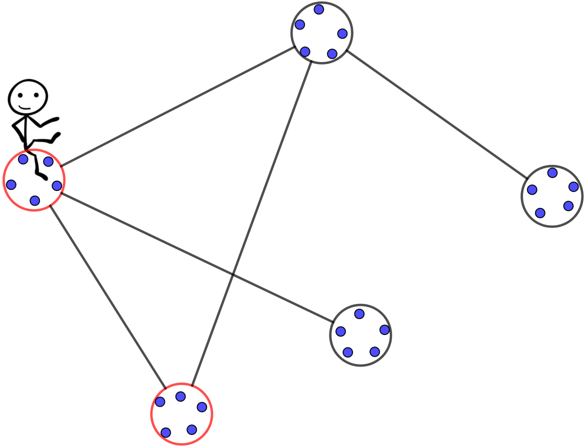

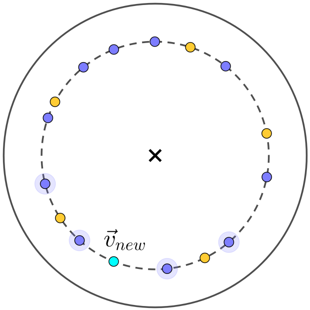

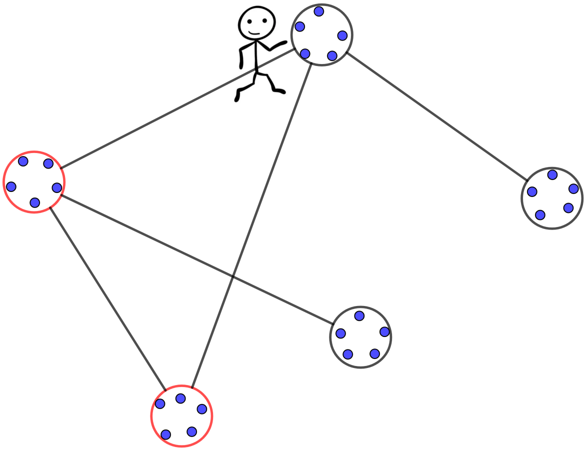

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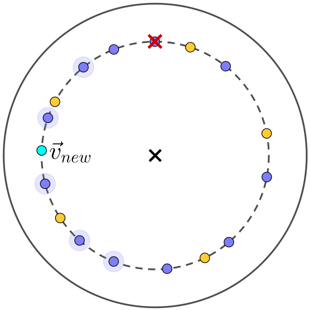

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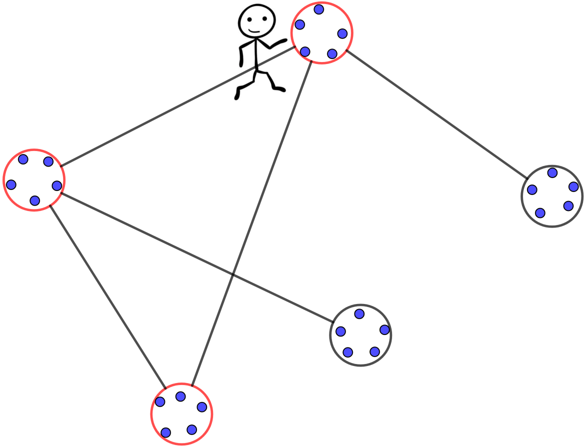

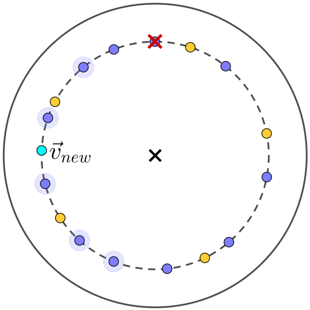

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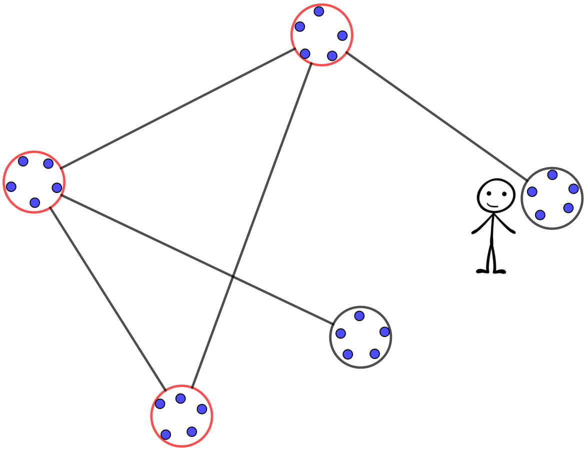

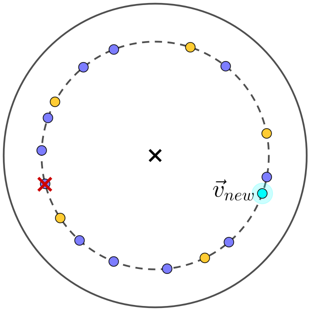

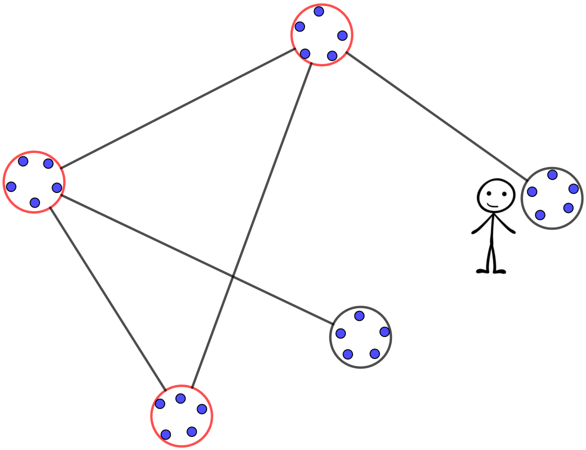

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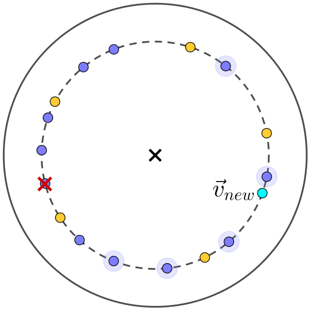

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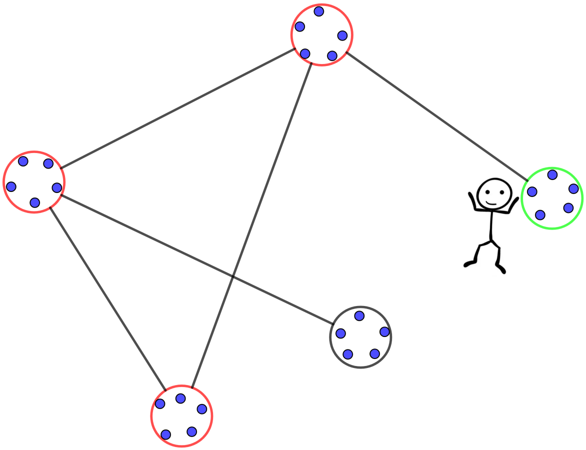

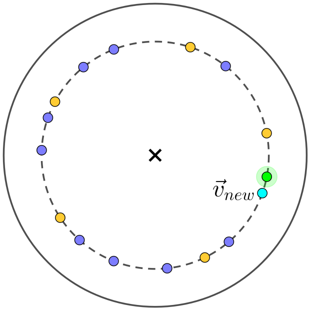

Classic VS Quantum Random Walk

Classic Random Walk

Randomly choose 1 neighbor vertex.

Quantum Random Walk

Quantum superposition of all the neighbors vertexes.

Step 1

Sample a code C and generate the α-filters.
Insert each list vector in its (unique) nearest α-filter. $\triangleright$ N^{c_α} vectors per α-filter

Step 2

For each α-filter :

1. VERTEX : Choose randomly N^{c_V} vectors from the α-filter. $\triangleright$ N^{c_V} vectors in the vertex
2. Sample a code C' and generate the β-filters.
   Insert each VERTEX's vector in its nearest β-filter.
3. Perform Quantum Random Walks to find all the reducing pairs in the α-filter.

Step 1

Sample a code C and generate the α-filters.
Insert each list vector in its (unique) nearest α-filter. ▷ N^{c_α} vectors per α-filter

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3. Perform Quantum Random Walks to find all the reducing pairs in the α-filter.

Repetitions

Run the steps 1-2 until we get N reduced vectors.

Complexity

Time complexity of a complete sieve step:

$$N \cdot \left(\mathcal{S} + \frac{1}{\sqrt{\epsilon}} \left(\frac{1}{\sqrt{\delta}} \mathcal{U} + \mathcal{C} \right) \right)$$

Parameters:

- c_α : N^{c_α} vectors per α-filter of C.
- c_V : N^{c_V} vectors per vertex in the graph.

Optimal complexity

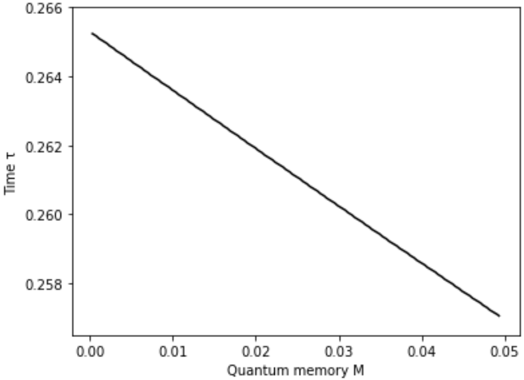
Our algorithm with parameters

$$c_\alpha \approx 0.3696 \quad ; \quad c_V \approx 0.2384$$

heuristically solves SVP on dimension d

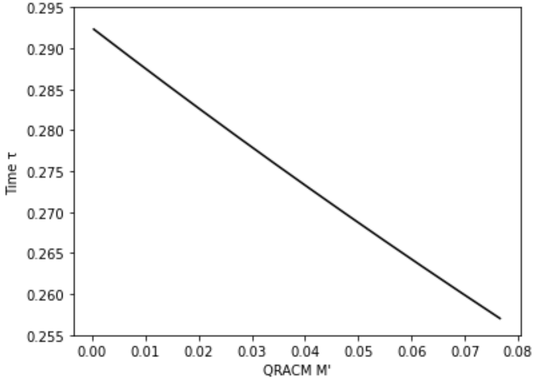
- in time $2^{0.2570d+o(d)}$,
- uses QRAM of maximum size $2^{0.0767d+o(d)}$,
- uses quantum memory of maximum size $2^{0.0495d+o(d)}$
- and uses classical memory of size $2^{0.2075d+o(d)}$.

Trade-off – fixed quantum memory



Quantum memory/time trade-off.

Trade-off – fixed QRAM



QRAM/time trade-off.

Trade-off – Synthesis

Time	**0.2925**	0.2827	0.2733	**0.2653**	0.2621	0.2598	**0.2570**
QRAM	**0**	0.02	0.04	**0.0578**	0.065	0.070	**0.0767**
Qmem	**0**	0	0	**0**	0.0190	0.0324	**0.0495**
Comment	[BDGL16] alg.			[Laa16] alg.			opt.param

Figure: Time, QRAM and quantum memory values for our algorithm.

Conclusion

- Time to break a cryptosystem based on SVP: $2^{0.2653d+o(d)} \to 2^{0.2570d+o(d)}$.
- 128 bits of security $\to$ 124.
- Fix with a slight increase of the parameters.

Thank you for your attention!

References

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